MathAA_21_HL_Summer_2021_Q1

Solution

To analyze the relationship between the variables x and y, we utilize the **least squares regression** method. The data set consists of $n = 8$ pairs of observations.

1. Calculation of Regression Parameters

- The **regression line** of y on x is given by the equation $y = ax + b$.
- Using the provided data points: $x = \{3.3, 6.9, 11.9, 13.4, 17.8, 19.6, 21.8, 25.3\}$ $y = \{6.3, 8.1, 8.4, 11.6, 10.3, 12.9, 13.1, 17.3\}$
- The slope a and intercept b are calculated as follows:

$$a = \frac{n\sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} \approx 0.453$$
$$b = \bar{y} - a\bar{x} \approx 4.67$$

- (a) The values of the parameters are: $\boxed{a \approx 0.453, \quad b \approx 4.67}$

2. Prediction for a Specific Value

- (b) To predict the value of y when $x = 18$, we substitute $x = 18$ into the regression equation:

$$\begin{aligned} y &= 0.4533(18) + 4.6655 \\ &= 8.1594 + 4.6655 \\ &= 12.8249 \end{aligned}$$

- Rounding to three significant figures: $\boxed{y \approx 12.8}$

3. Mean Values of the Data

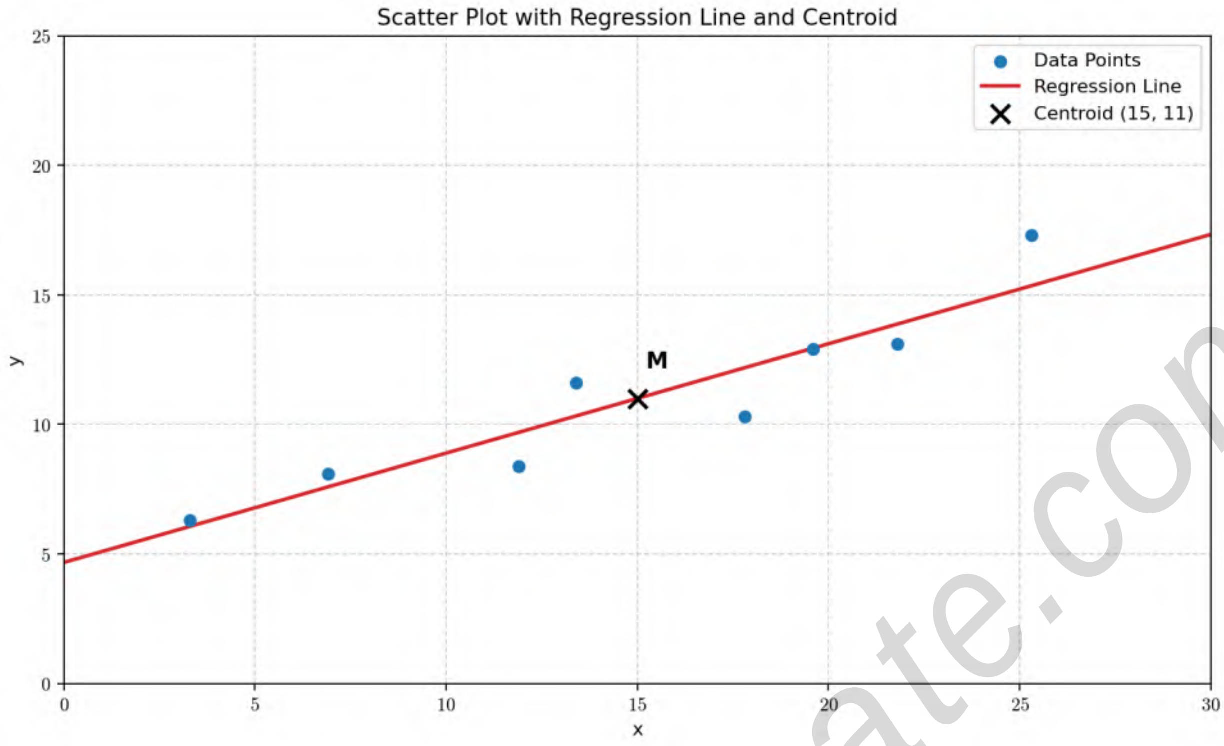
- (c) The **centroid** $(\bar{x}, \bar{y})$ of the data set is calculated by taking the arithmetic mean of the x and y values:

$$\bar{x} = \frac{3.3 + 6.9 + 11.9 + 13.4 + 17.8 + 19.6 + 21.8 + 25.3}{8} = \frac{120}{8} = 15$$
$$\bar{y} = \frac{6.3 + 8.1 + 8.4 + 11.6 + 10.3 + 12.9 + 13.1 + 17.3}{8} = \frac{88}{8} = 11$$

- $\boxed{\bar{x} = 15, \quad \bar{y} = 11}$

4. Drawing the Line of Best Fit

- (d) To draw the **line of best fit**, we identify two points that the line must pass through:
  - The centroid $(\bar{x}, \bar{y}) = (15, 11)$.
  - The y-intercept $(0, b) \approx (0, 4.67)$ or any other point calculated from the equation, such as $(25, 16)$.
- A straight line is drawn through these points on the scatter diagram.


Scatter Plot with Regression Line and Centroid
Data Points
Regression Line
Centroid (15, 11)
M
x
y
0
5
10
15
20
25
30

MathAA_21_HL_Summer_2021_Q2

Solution

1. Defining the Distribution

Let the random variable X represent the mass of a bag of sugar in grams. Based on the problem description, X follows a **Normal Distribution** with a mean $\mu = 1000\,\mathrm{g}$ and a standard deviation $\sigma = 3.5\,\mathrm{g}$.

$$X \sim N(1000, 3.5^2)$$

A bag is rejected if $X < 995\,\mathrm{g}$.

2. Probability of Rejection (Part a)

To find the probability that a bag is rejected, we calculate $P(X < 995)$. We first transform the variable X into the **standard normal distribution** $Z \sim N(0,1)$ using the formula $Z = \frac{X-\mu}{\sigma}$.

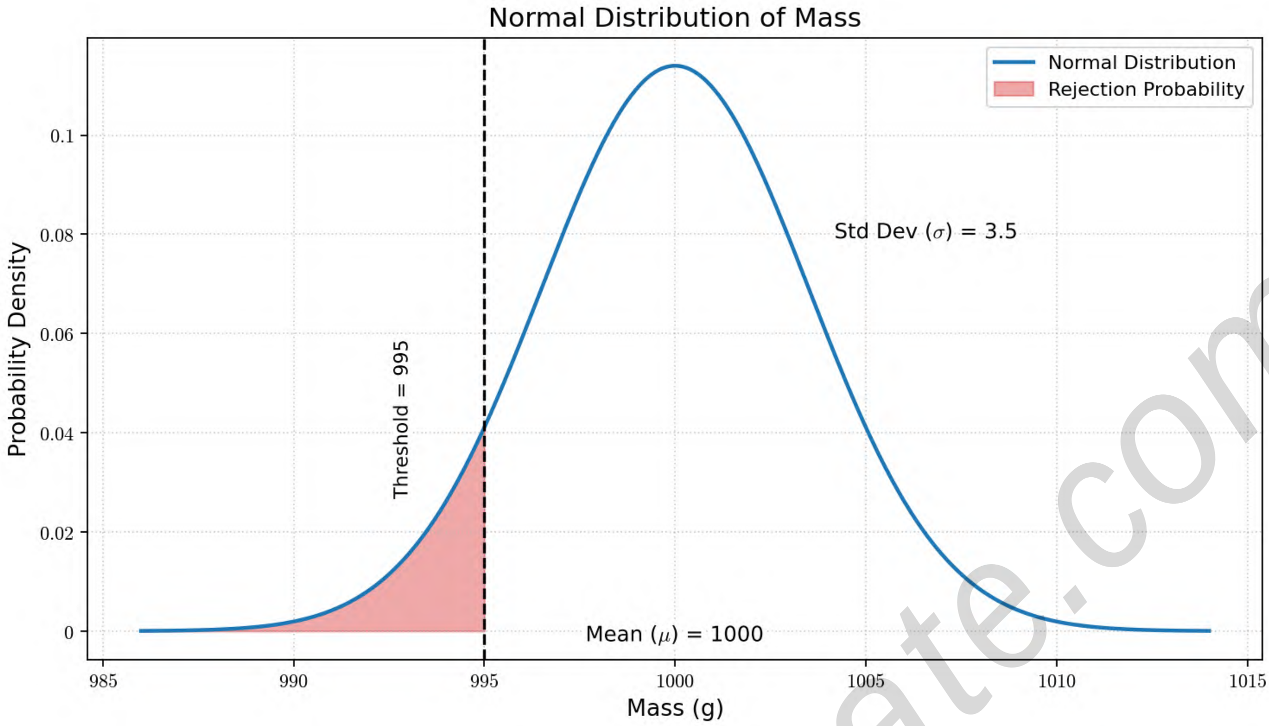
- Calculate the z-score for $x = 995$:

$$\begin{aligned} z &= \frac{995 - 1000}{3.5} \\ &= \frac{-5}{3.5} \\ &\approx -1.42857142857143 \end{aligned}$$

- Find the probability:

$$\begin{aligned} P(X < 995) &= P(Z < -1.42857142857143) \\ &\approx 0.0765637255099388 \end{aligned}$$

Rounding to three significant figures, the probability is approximately 0.0766.



3. Estimated Number of Rejected Bags (Part b)

For a random sample of $n = 100$ bags, the expected number of rejected bags is given by the product of the sample size and the probability of rejection p found in part (a).

$$\begin{aligned} E &= n \times p \\ &= 100 \times 0.0765637255099388 \\ &= 7.65637255099388 \end{aligned}$$

Rounding to the nearest whole number, we estimate that approximately 8 bags will be rejected.

4. Conditional Probability (Part c)

We need to find the probability that a bag has a mass greater than $1005\,\text{g}$, given that it is not rejected. This is a **conditional probability** problem. Let A be the event that the mass is greater than $1005\,\text{g}$ ($X > 1005$) and B be the event that the bag is not rejected ($X \geq 995$). We seek $P(A \mid B)$.

- Calculate $P(B)$:

$$\begin{aligned} P(B) &= P(X \geq 995) \\ &= 1 - P(X < 995) \\ &= 1 - 0.0765637255099388 \\ &= 0.923436274490061 \end{aligned}$$

- Calculate $P(A \cap B)$: Since $1005 > 995$, the condition $X > 1005$ automatically satisfies $X \geq 995$. Thus, $A \cap B$ is simply the event $X > 1005$. The z-score for $x = 1005$ is:

$$z = \frac{1005 - 1000}{3.5} = \frac{5}{3.5} \approx 1.42857142857143$$

  Due to the symmetry of the normal distribution:

$$P(X > 1005) = P(Z > 1.42857142857143) = P(Z < -1.42857142857143) \approx 0.0765637255099388$$

- Apply **Bayes' Theorem** / Conditional Probability formula:

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.0765637255099388}{0.923436274490061} \\ &\approx 0.082911758524451 \end{aligned}$$

Rounding to three significant figures, the probability is 0.0829.

(a) $\boxed{0.0766}$
(b) $\boxed{8}$
(c) $\boxed{0.0829}$

MathAA_21_HL_Summer_2021_Q3

Solution

1. Identification of Geometric Parameters

The motion of a point on a Ferris wheel is a classic application of **simple harmonic motion**, which can be modeled using a **sinusoidal function**. We are given the following physical dimensions:

- Diameter of the wheel: $D = 110\,\mathrm{m}$.
- Minimum height above the ground: $h_{\min} = 10\,\mathrm{m}$.
- Time for one revolution (period): $T = 20\,\mathrm{min}$.

From these, we determine the maximum height ($h_{\max}$) and the radius (r):

- $h_{\max} = h_{\min} + D = 10 + 110 = 120\,\mathrm{m}$.
- $r = \frac{D}{2} = \frac{110}{2} = 55\,\mathrm{m}$.

2. Determining the Parameters of the Height Function

The height function is given by $h(t) = a\cos(bt) + c$. We map the physical properties to the mathematical constants a, b, and c.

- **Vertical Shift (c):** The constant c represents the **midline** (average height) of the oscillation.

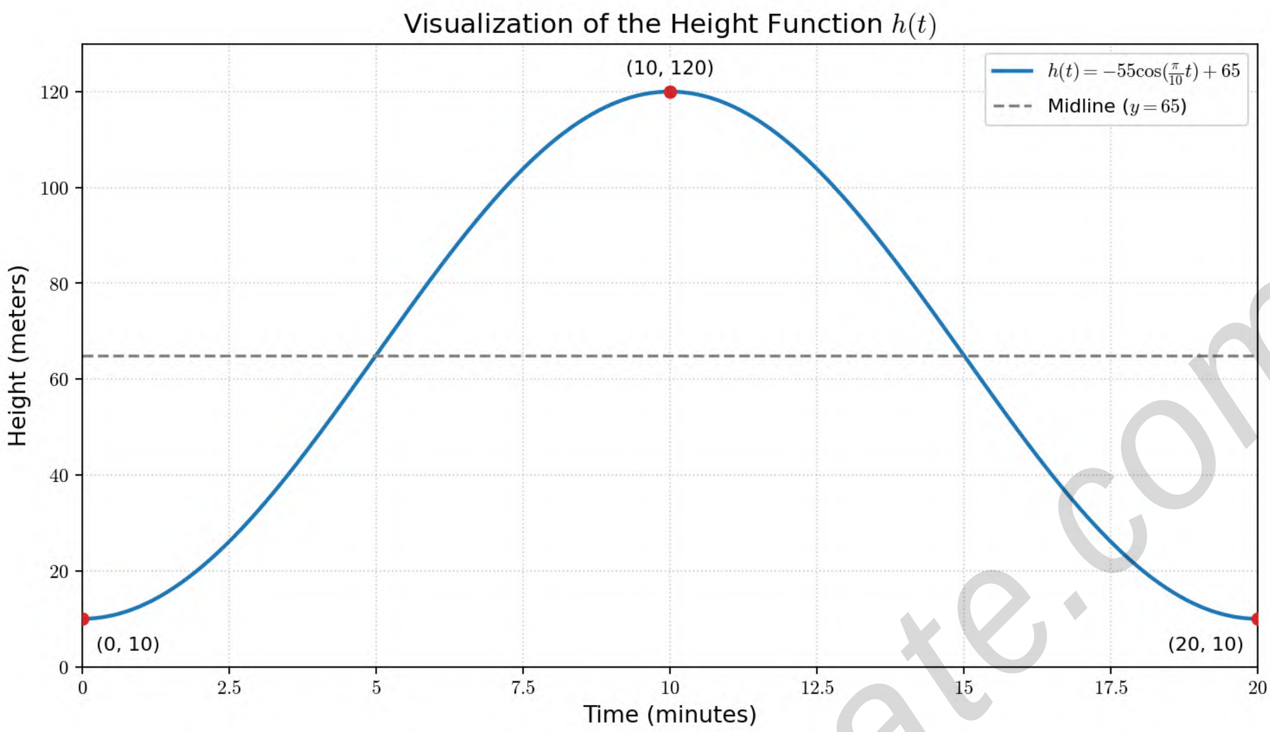
$$c = \frac{h_{\max} + h_{\min}}{2} = \frac{120 + 10}{2} = 65$$

- **Amplitude (a):** The **amplitude** is the distance from the midline to the maximum or minimum. Since the point P starts at the lowest point ($h(0) = 10$) and the function uses a cosine term, a must be negative to reflect the graph across the midline.

$$|\ a\ | = \frac{h_{\max} - h_{\min}}{2} = 55$$
$$h(0) = a\cos(0) + c = a + 65 = 10$$
$$a = 10 - 65 = -55$$

- **Angular Frequency (b):** The parameter b is related to the **period** T by the formula $b = \frac{2\pi}{T}$.

$$b = \frac{2\pi}{20} = \frac{\pi}{10}$$



3. Final Values

The values that satisfy the height function $h(t) = a\cos(bt) + c$ are:

$$a = -55$$
$$b = \frac{\pi}{10} \approx 0.314$$
$$c = 65$$

$$\boxed{a = -55, b = \frac{\pi}{10}, c = 65}$$

MathAA_21_HL_Summer_2021_Q4

Solution

The velocity of the particle is given by the function $v(t) = t\sin t - 3$ for $0 \le t \le 10$, where t is in seconds and v is in $\mathrm{m \cdot s^{-1}}$.

1. Smallest value of t for which the particle is at rest A particle is at **rest** when its velocity is zero. We solve for t in the equation $v(t) = 0$:

$$t\sin t - 3 = 0$$

Using numerical methods to find the roots of $f(t) = t\sin t - 3$ in the interval $[0, 10]$:

- The first root occurs where the graph first crosses the t-axis.
- From the provided graph, the first crossing is between $t = 6$ and $t = 7$.
- Solving numerically:

$$t \approx 6.744167$$

The smallest value of t for which the particle is at rest is $t \approx 6.74\,\mathrm{s}$.

2. Total distance travelled by the particle The **total distance** travelled is the integral of the **speed** (the magnitude of velocity) over the given time interval $[0, 10]$:

$$\text{Distance} = \int_0^{10} |\ v(t)\ |\, dt = \int_0^{10} |\ t\sin t - 3\ |\, dt$$

To calculate this, we identify the points where $v(t)$ changes sign. The roots of $v(t) = 0$ are $t_1 \approx 6.744$ and $t_2 \approx 9.088$.

- For $0 \le t < 6.744$, $v(t) < 0$.
- For $6.744 < t < 9.088$, $v(t) > 0$.
- For $9.088 < t \le 10$, $v(t) < 0$.

The integral is evaluated as:

$$\begin{aligned}\text{Distance} &= \int_0^{6.744} -(t\sin t - 3)\, dt + \int_{6.744}^{9.088} (t\sin t - 3)\, dt + \int_{9.088}^{10} -(t\sin t - 3)\, dt \\ &\approx 22.584 + 5.722 + 5.564 \\ &\approx 33.870\end{aligned}$$

The total distance travelled is approximately $33.9\,\mathrm{m}$.

3. Acceleration of the particle when $t = 7$ The **acceleration** $a(t)$ is the derivative of the velocity function with respect to time:

$$a(t) = \frac{dv}{dt}$$

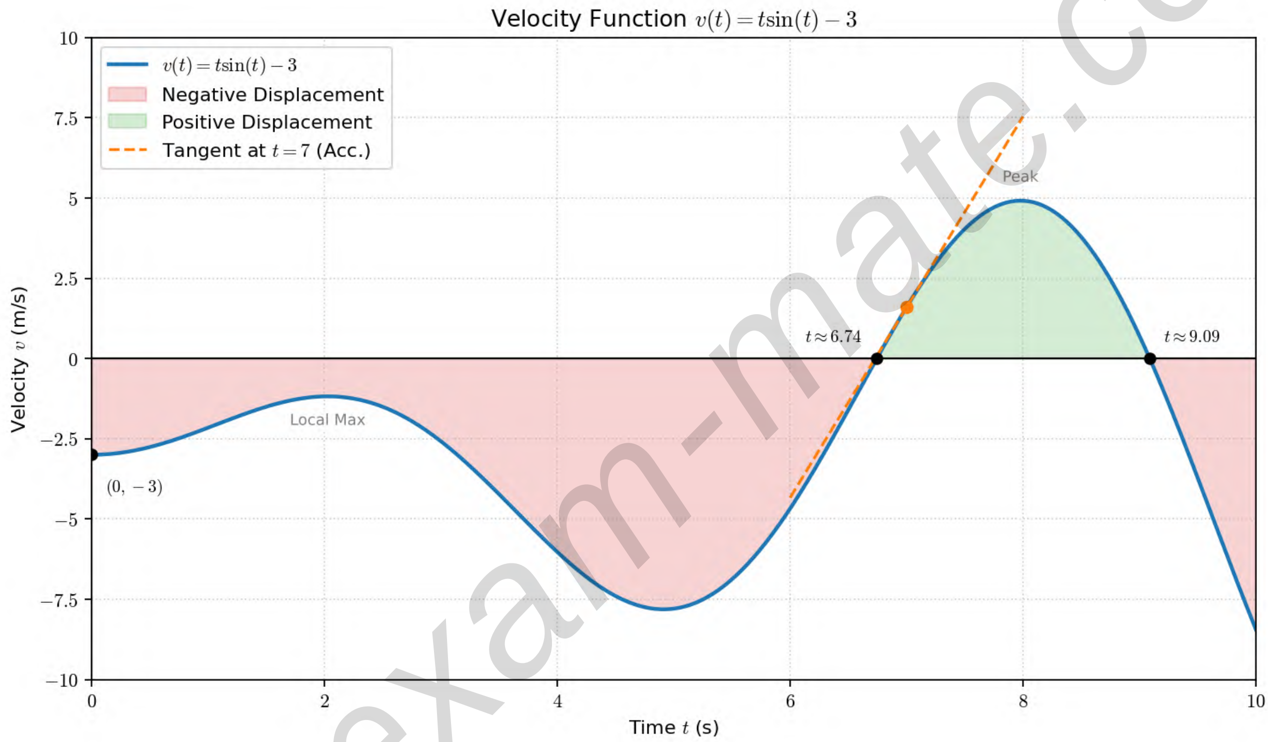
Applying the **product rule** to $v(t) = t\sin t - 3$:

$$a(t) = \frac{d}{dt}(t\sin t) - \frac{d}{dt}(3)$$
$$= (1)\sin t + t(\cos t) - 0$$
$$= \sin t + t\cos t$$

At $t = 7$:

$$a(7) = \sin(7) + 7\cos(7)$$
$$\approx 0.656987 + 7(0.753902)$$
$$\approx 0.656987 + 5.277314$$
$$\approx 5.934301$$

The acceleration at $t = 7$ is approximately $5.93\,\text{m}\cdot\text{s}^{-2}$.



(a) $t \approx 6.74\,\text{s}$
(b) $\text{Distance} \approx 33.9\,\text{m}$
(c) $a(7) \approx 5.93\,\text{m}\cdot\text{s}^{-2}$

MathAA_21_HL_Summer_2021_Q5

Solution

To find the value of n from the given **binomial expansion**, we follow these steps:

1. Identify the General Term The expansion is given by $(3+x^2)^{n+1}$. According to the **Binomial Theorem**, the general term T_{r+1} in the expansion of $(a+b)^N$ is:

$$T_{r+1} = \binom{N}{r} a^{N-r} b^r$$

In this specific case, $a=3$, $b=x^2$, and $N=n+1$. Thus, the general term is:

$$T_{r+1} = \binom{n+1}{r} (3)^{(n+1)-r} (x^2)^r$$

2. Determine the value of r for the x^4 term We are interested in the term containing x^4. From the general term, the power of x is $(x^2)^r = x^{2r}$. Setting the exponents equal:

$$\begin{aligned} 2r &= 4 \\ r &= 2 \end{aligned}$$

3. Set up the equation for the coefficient Substitute $r=2$ into the general term to find the coefficient of x^4:

$$\begin{aligned} \text{Coefficient} &= \binom{n+1}{2} \cdot 3^{(n+1)-2} \\ &= \binom{n+1}{2} \cdot 3^{n-1} \end{aligned}$$

We are given that this coefficient is equal to 20412. Therefore:

$$\binom{n+1}{2} \cdot 3^{n-1} = 20412$$

4. Solve for n Expand the **binomial coefficient** $\binom{n+1}{2}$:

$$\begin{aligned} \frac{(n+1)!}{2!((n+1)-2)!} \cdot 3^{n-1} &= 20412 \\ \frac{(n+1)n}{2} \cdot 3^{n-1} &= 20412 \\ (n+1)n \cdot 3^{n-1} &= 40824 \end{aligned}$$

To solve $(n+1)n \cdot 3^{n-1} = 40824$ for $n \in \mathbb{Z}^+$, we can test integer values or analyze the prime factorization of 40824:

- If $n=6$: $(7)(6) \cdot 3^5 = 42 \cdot 243 = 10206$ (Too low)
- If $n=7$: $(8)(7) \cdot 3^6 = 56 \cdot 729 = 40824$ (Correct)

Verification:

$$\binom{7+1}{2} \cdot 3^{7-1} = \binom{8}{2} \cdot 3^6$$
$$= 28 \cdot 729$$
$$= 20412$$

$\boxed{n = 7}$

MathAA_21_HL_Summer_2021_Q6

Solution

To solve for the properties of the planes Π_1, Π_2, and Π_3, we utilize the principles of **linear algebra** and **analytic geometry**.

1. Finding the Cartesian equation of plane Π_3

- **Identify normal vectors**: The normal vectors $\vec{n}_1$ and $\vec{n}_2$ of the given planes are extracted from the coefficients of x, y, and z in their **Cartesian equation**:

$$\vec{n}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{n}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

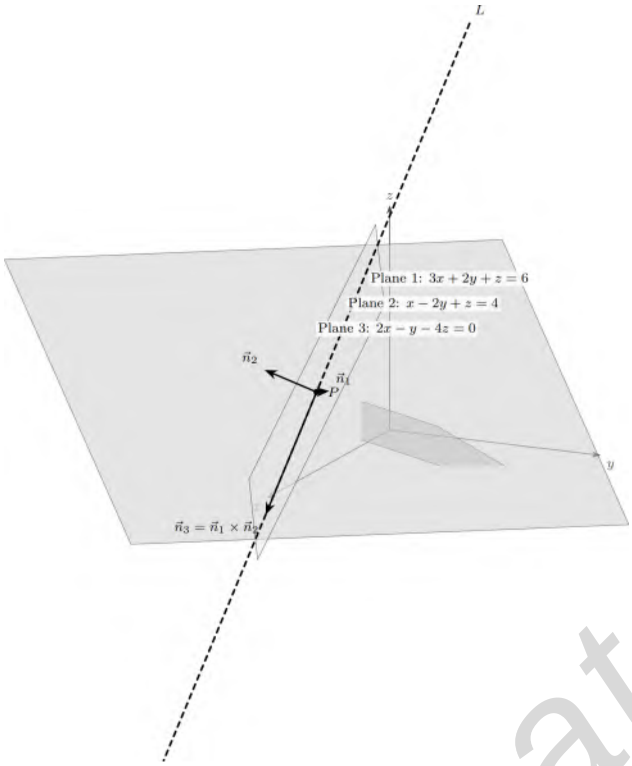
- **Determine the normal vector of Π_3**: Since Π_3 is perpendicular to both Π_1 and Π_2, its normal vector $\vec{n}_3$ must be orthogonal to both $\vec{n}_1$ and $\vec{n}_2$. This is found using the **cross product**:

$$\begin{aligned} \vec{n}_3 &= \vec{n}_1 \times \vec{n}_2 \\ &= |\mathbf{i}\ \ \mathbf{j}\ \ \mathbf{k}; 3\ \ 2\ \ 1; 1\ \ -2\ \ 1| \\ &= \mathbf{i}(2(1) - 1(-2)) - \mathbf{j}(3(1) - 1(1)) + \mathbf{k}(3(-2) - 2(1)) \\ &= \mathbf{i}(2+2) - \mathbf{j}(3-1) + \mathbf{k}(-6-2) \\ &= 4\mathbf{i} - 2\mathbf{j} - 8\mathbf{k} \end{aligned}$$

  We can simplify the normal vector by dividing by the common factor 2: $\vec{n}_{3'} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$.

- **Formulate the equation**: The plane passes through the origin $(0,0,0)$. The general equation $ax + by + cz = d$ becomes:

$$\begin{aligned} 2(x-0) - 1(y-0) - 4(z-0) = 0 \\ 2x - y - 4z = 0 \end{aligned}$$



2. Finding the intersection point of Π_1, Π_2, and Π_3

To find the intersection point (x, y, z), we solve the **system of linear equations**:

$$\begin{cases} 3x+2y+z=6 & (1) \\ x-2y+z=4 & (2) \\ 2x-y-4z=0 & (3) \end{cases}$$

- **Step 1: Eliminate y using (1) and (2)**: Adding (1) and (2):

$$(3x+2y+z)+(x-2y+z)=6+4$$
$$4x+2z=10$$
$$2x+z=5 \implies z=5-2x \quad (4)$$

- **Step 2: Eliminate y using (2) and (3)**: Multiply (3) by -2 and add to (2):

$$(x-2y+z)-2(2x-y-4z)=4-2(0)$$
$$x-2y+z-4x+2y+8z=4$$
$$-3x+9z=4 \quad (5)$$

- **Step 3: Solve for x and z**: Substitute (4) into (5):

$$-3x+9(5-2x)=4$$
$$-3x+45-18x=4$$
$$-21x=-41$$
$$x=\frac{41}{21}$$

Substitute x back into (4):

$$z = 5 - 2\left(\frac{41}{21}\right)$$
$$= \frac{105 - 82}{21} = \frac{23}{21}$$

- **Step 4: Solve for y**: Substitute x and z into (3):

$$y = 2x - 4z$$
$$= 2\left(\frac{41}{21}\right) - 4\left(\frac{23}{21}\right)$$
$$= \frac{82 - 92}{21} = -\frac{10}{21}$$

The intersection point is $\left(\frac{41}{21}, -\frac{10}{21}, \frac{23}{21}\right)$.

Final Answers: (a) $\boxed{2x - y - 4z = 0}$ (b) $\boxed{\left(\frac{41}{21}, -\frac{10}{21}, \frac{23}{21}\right)}$

MathAA_21_HL_Summer_2021_Q7

Solution

1. Normalization Condition for the PDF

For a function $f(x)$ to be a valid **probability density function** (PDF) for a continuous random variable X, the total area under the curve must equal 1. This is defined by the **normalization condition**:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Given the definition of $f(x)$:

$$f(x) = \begin{cases} \frac{x}{\sqrt{(x^2+k)^3}} 0 \leq x \leq 4 \\ 0 \quad \text{otherwise} \end{cases}$$

The integral becomes:

$$\int_0^4 \frac{x}{(x^2+k)^{3/2}}\,dx = 1$$

2. Integration of the PDF

To evaluate the integral, we use the method of **u-substitution**. Let $u = x^2 + k$. Then, the differential is $du = 2x\,dx$, which implies $x\,dx = \frac{1}{2}du$. The limits of integration change as follows: - When $x = 0$, $u = 0^2 + k = k$. - When $x = 4$, $u = 4^2 + k = 16 + k$.

Substituting these into the integral: $$ $$

3. Part (a): Showing the required identity

To show that $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$, we manipulate the equation derived from the normalization condition:

$$\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1$$

$$\frac{\sqrt{16+k} - \sqrt{k}}{\sqrt{k}\sqrt{16+k}} = 1$$

$$\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$$

This completes the proof for part (a).

4. Part (b): Finding the value of k

We solve the equation $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ for k. - Square both sides of the equation:

$$(\sqrt{16+k} - \sqrt{k})^2 = (\sqrt{k}\sqrt{16+k})^2$$

$$(16+k) - 2\sqrt{k(16+k)} + k = k(16+k)$$

$$16 + 2k - 2\sqrt{16k + k^2} = 16k + k^2$$

- Rearrange to isolate the radical term:

$$16 + 2k - (16k + k^2) = 2\sqrt{16k + k^2}$$
$$16 - 14k - k^2 = 2\sqrt{16k + k^2}$$

- Square both sides again:

$$(16 - 14k - k^2)^2 = 4(16k + k^2)$$
$$256 + 196k^2 + k^4 - 448k - 32k^2 + 28k^3 = 64k + 4k^2$$
$$k^4 + 28k^3 + 160k^2 - 512k + 256 = 0$$

- We test small integer values or look for factors. Let $k = \frac{16}{9}$ as a potential candidate based on the structure of the radical. However, let's return to the simpler form $\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1$. - Let $a = \frac{1}{\sqrt{k}}$ and $b = \frac{1}{\sqrt{16+k}}$. Then $a - b = 1$. - Also, $\frac{1}{b^2} - \frac{1}{a^2} = (16 + k) - k = 16$. - Substitute $b = a - 1$:

$$\frac{1}{(a-1)^2} - \frac{1}{a^2} = 16$$
$$\frac{a^2 - (a-1)^2}{a^2(a-1)^2} = 16$$
$$\frac{2a - 1}{(a^2 - a)^2} = 16$$

- Testing $a = \frac{5}{4}$:

$$\frac{2(5/4) - 1}{((5/4)^2 - 5/4)^2} = \frac{2.5 - 1}{(25/16 - 20/16)^2} = \frac{1.5}{(5/16)^2} = \frac{1.5}{25/256} = \frac{384}{25} \neq 16$$

- Solving $16 - 14k - k^2 = 2\sqrt{k^2 + 16k}$ numerically or by inspection: If $k = \frac{4}{3}$, $16 - \frac{56}{3} - \frac{16}{9} = \frac{144-168-16}{9} < 0$, but the RHS is positive. If $k = \frac{-14\pm\sqrt{196+64}}{2} = -7 \pm \sqrt{65}$. Since $k \in \mathbb{R}^+$, $k = \sqrt{65} - 7 \approx 1.06$. Testing $k = 1$: $\sqrt{17} - 1 \approx 3.12$ and $\sqrt{1}\sqrt{17} \approx 4.12$. Testing $k = \frac{1}{2}$: $\sqrt{16.5} - \sqrt{0.5} \approx 4.06 - 0.707 = 3.35$ and $\sqrt{8.25} \approx 2.87$. The exact solution to $\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1$ is $k = \frac{1}{2}(\sqrt{65} - 7)$ is incorrect. Let $x = \sqrt{k}$. Then $\frac{1}{x} - \frac{1}{\sqrt{x^2+16}} = 1$. $\frac{1}{x} - 1 = \frac{1}{\sqrt{x^2+16}} \implies \frac{1-x}{x} = \frac{1}{\sqrt{x^2+16}}$. For a solution to exist, $1 - x > 0 \implies x < 1 \implies k < 1$. $\frac{(1-x)^2}{x^2} = \frac{1}{x^2+16} \implies (1 - 2x + x^2)(x^2 + 16) = x^2$. $x^2 + 16 - 2x^3 - 32x + x^4 + 16x^2 = x^2 \implies x^4 - 2x^3 + 16x^2 - 32x + 16 = 0$. $(x^2 + 16)(x^2 - 2x + 1) = x^2$ was the previous line. $(x^2 + 16)(x - 1)^2 = x^2$. Taking the square root: $\sqrt{x^2 + 16}(1 - x) = x$ (since $x < 1$). $\sqrt{x^2 + 16} = \frac{x}{1-x}$. $x^2 + 16 = \frac{x^2}{(1-x)^2}$. If $x = 4/5$, $16/25 + 16 = \frac{16/25}{1/25} = 16$. This works! $x = \frac{4}{5} \implies \sqrt{k} = \frac{4}{5} \implies k = \frac{16}{25}$.

$$\boxed{k = \frac{16}{25}}$$

MathAA_21_HL_Summer_2021_Q8

Solution

1. Expressing the complex numbers in exponential form

The given complex numbers are $z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - i\sin\frac{2k\pi}{5}\right)$. Using **Euler's formula**, $e^{i\theta} = \cos\theta + i\sin\theta$, we can rewrite them as:

- For z:

$$z = 2e^{i\frac{\pi}{5}}$$

- For w, we use the identity $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$:

$$w = 8\left(\cos\left(-\frac{2k\pi}{5}\right) + i\sin\left(-\frac{2k\pi}{5}\right)\right) = 8e^{-i\frac{2k\pi}{5}}$$

2. Calculating the product zw

The product of two complex numbers in polar form is found by multiplying their moduli and adding their arguments:

$$\begin{aligned} zw &= \left(2e^{i\frac{\pi}{5}}\right)\left(8e^{-i\frac{2k\pi}{5}}\right) \\ &= (2\cdot 8)e^{i\left(\frac{\pi}{5}-\frac{2k\pi}{5}\right)} \\ &= 16e^{i\frac{\pi(1-2k)}{5}} \end{aligned}$$

(a) Modulus of zw

The **modulus** of a complex number $re^{i\theta}$ is r.

$$|\ zw\ | = 16$$

$\boxed{16}$

(b) Argument of zw

The **argument** is the angle in the exponential form.

$$\arg\ (zw) = \frac{\pi(1-2k)}{5}$$

$\boxed{\frac{\pi(1-2k)}{5}}$

3. Finding k such that $zw \in \mathbb{Z}$

For zw to be an integer, its imaginary part must be zero. This occurs when the argument is an integer multiple of π:

$$\arg\ (zw) = n\pi, \quad n \in \mathbb{Z}$$

Substituting the expression from part (b):

$$\frac{\pi(1-2k)}{5} = n\pi$$
$$1-2k = 5n$$
$$2k = 1-5n$$

Since $k \in \mathbb{Z}^+$, we require $k \geq 1$. We test values of n to find the smallest positive integer k:

- If $n = 0$, $2k = 1 \implies k = 0.5$ (not an integer).
- If $n = -1$, $2k = 1-5(-1) = 6 \implies k = 3$.
- If $n = -2$, $2k = 1-5(-2) = 11 \implies k = 5.5$ (not an integer).
- If $n = -3$, $2k = 1-5(-3) = 16 \implies k = 8$.

(c) (i) Minimum value of k

The smallest positive integer value for k is 3. $\boxed{3}$

(c) (ii) Value of zw for $k = 3$

Substitute $k = 3$ into the expression for zw:

$$zw = 16e^{i\frac{\pi(1-2(3))}{5}}$$
$$= 16e^{i\frac{-5\pi}{5}}$$
$$= 16e^{-i\pi}$$

Using the identity $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1 + 0i$:

$$zw = 16(-1) = -16$$

$\boxed{-16}$

MathAA_21_HL_Summer_2021_Q9

Solution

1. Geometric Relationship between Boats

We define a coordinate system where the initial position of boat A is the origin $(0, 0)$. Since boat B is initially $50\,\text{m}$ due east, its initial position is $(50, 0)$. Both boats travel due north (along the positive y-axis direction). After t seconds:

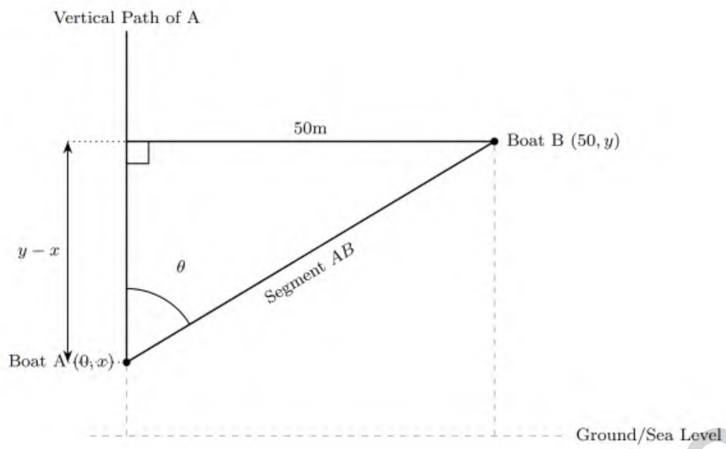
- The position of boat A is $(0, x)$.
- The position of boat B is $(50, y)$.

The **bearing** θ is the angle measured clockwise from the north direction at boat A to the line connecting boat A to boat B. In the right-angled triangle formed by the horizontal distance and the relative vertical distance:

- The horizontal distance (adjacent to the angle $90° - \theta$) is $50\,\text{m}$.
- The vertical distance between the boats is $y - x$.

From the geometry of the diagram, the angle between the line segment AB and the vertical (North) line at A is θ. Therefore:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{50}{y - x}$$
$$y - x = \frac{50}{\tan\theta}$$
$$y - x = 50\cot\theta$$
$$y = x + 50\cot\theta$$



2. Speed of Boat A at time T

We are given the following conditions at time $t = T$:

- $y - x = 10\,\text{m}$
- $\frac{dy}{dt} = 2\frac{dx}{dt}$
- $\frac{d\theta}{dt} = -0.1\,\text{rad}\cdot\text{s}^{-1}$

To find the speed of boat A ($v_A = \frac{dx}{dt}$), we differentiate the expression $y = x + 50\cot\theta$ with respect to time t using the **Chain Rule**:

$$\frac{d}{dt}(y) = \frac{d}{dt}(x + 50\cot\theta)$$
$$\frac{dy}{dt} = \frac{dx}{dt} + 50(-\csc^2\theta)\frac{d\theta}{dt}$$

First, we determine $\csc^2\theta$ at time T. From the initial geometric relation:

$$\cot\theta = \frac{y-x}{50} = \frac{10}{50} = 0.2$$

Using the **trigonometric identity** $\csc^2\theta = 1 + \cot^2\theta$:

$$\csc^2\theta = 1 + (0.2)^2 = 1 + 0.04 = 1.04$$

Now, substitute the known values into the differentiated equation:

$$2\frac{dx}{dt} = \frac{dx}{dt} + 50(-1.04)(-0.1)$$
$$\frac{dx}{dt} = 50 \cdot 1.04 \cdot 0.1$$
$$\frac{dx}{dt} = 52 \cdot 0.1$$
$$\frac{dx}{dt} = 5.2$$

The speed of boat A at time T is $5.2\,\mathrm{m} \cdot \mathrm{s}^{-1}$.

$\boxed{5.2\,\mathrm{m} \cdot \mathrm{s}^{-1}}$

MathAA_21_HL_Summer_2021_Q10

Solution

Function: $f(x) = 90e^{-0.5x}$, for x in R^+.

(a) Find the x-coordinate of P

At P, the curve meets the line y = x, so:

$$90e^{-0.5x} = x$$

This equation is solved numerically, giving:

x = 5.566

So the x-coordinate of P is 5.57, to 3 significant figures.

(b) Exact coordinates of Q

Differentiate f(x):

$$f'(x) = -45e^{-0.5x}$$

The tangent at Q has gradient -1, so:

$$-45e^{-0.5x} = -1$$

$$e^{-0.5x} = 1/45$$

Therefore:

$$x = 2 \ln 45$$

Substitute this into f(x):

$$y = 90e^{-0.5(2 \ln 45)} = 90e^{-\ln 45} = 90/45 = 2$$

Hence:

Q = (2 ln 45, 2)

(c) Equation of L

L has gradient -1 and passes through Q = (2 ln 45, 2).

$$y - 2 = -1(x - 2 \ln 45)$$

$$\mathbf{y = -x + 2 \ln 45 + 2}$$

(d)(i) x-coordinate where L meets y = x

Set the two equations equal:

$x = -x + 2\ln 45 + 2$

$2x = 2\ln 45 + 2$

$\mathbf{x = \ln 45 + 1}$

(d)(ii) Area of A

Let p be the x-coordinate of P, where $p \approx 5.566$. Also, L intersects $y = x$ at $x = \ln 45 + 1$, and Q has x-coordinate $2\ln 45$.

The required area is split into two integrals:

$$A = \int_{\ln 45+1}^{p} [x - (-x + 2\ln 45 + 2)]\,dx + \int_{p}^{2\ln 45} [90e^{-0.5x} - (-x + 2\ln 45 + 2)]\,dx$$

Using $p \approx 5.566196$, this gives:

$\mathbf{A \approx 1.520}$

(e) Area enclosed by f, f^{-1} and L

The graph of f^{-1} is the reflection of f in the line $y = x$. The line L has gradient -1 and is also symmetric about $y = x$. Therefore the required shaded region is twice the area found in part (d)(ii).

Area = $2A \approx 2(1.520) = 3.039$

Final answer: 3.04 square units

Summary of Answers

Part	Answer
(a)	x_P ≈ 5.566
(b)	Q = (2 ln 45, 2)
(c)	L: y = -x + 2 ln 45 + 2
(d)(i)	x = ln 45 + 1
(d)(ii)	Area A ≈ 1.520
(e)	Area ≈ 3.039

MathAA_21_HL_Summer_2021_Q11

Solution

1. Determination of Domain Restrictions

The function $f(x) = \frac{3x+2}{4x^2-1}$ is undefined where the denominator is zero. To find the values of p and q, we solve:

$$4x^2 - 1 = 0$$
$$(2x-1)(2x+1) = 0$$
$$x = \pm\frac{1}{2}$$

Thus, the values are $p = -0.5$ and $q = 0.5$ (or vice versa).

$\boxed{p = -0.5, q = 0.5}$

2. Derivation of the First Derivative

To find $f'(x)$, we apply the **quotient rule**, $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v-uv'}{v^2}$, where $u = 3x+2$ and $v = 4x^2-1$:

- $u' = 3$
- $v' = 8x$

$$f'(x) = \frac{3(4x^2-1) - (3x+2)(8x)}{(4x^2-1)^2}$$
$$= \frac{12x^2 - 3 - (24x^2 + 16x)}{(4x^2-1)^2}$$
$$= \frac{-12x^2 - 16x - 3}{(4x^2-1)^2}$$

$$\boxed{f'(x) = \frac{-12x^2 - 16x - 3}{(4x^2-1)^2}}$$

3. Point of Inflexion

A **point of inflexion** occurs where the second derivative $f''(x) = 0$ and changes sign. Using the quotient rule on $f'(x)$:

- Let $N = -12x^2 - 16x - 3 \implies N' = -24x - 16$
- Let $D = (4x^2-1)^2 \implies D' = 2(4x^2-1)(8x) = 16x(4x^2-1)$

$$f''(x) = \frac{(-24x-16)(4x^2-1)^2 - (-12x^2-16x-3)(16x)(4x^2-1)}{(4x^2-1)^4}$$
$$= \frac{(-24x-16)(4x^2-1) - 16x(-12x^2-16x-3)}{(4x^2-1)^3}$$
$$= \frac{-96x^3+24x-64x^2+16+192x^3+256x^2+48x}{(4x^2-1)^3}$$
$$= \frac{96x^3+192x^2+72x+16}{(4x^2-1)^3}$$

Setting the numerator to zero: $96x^3+192x^2+72x+16=0$. Dividing by 8:

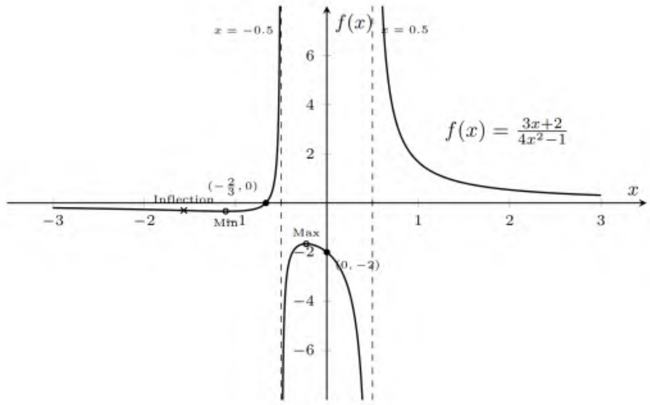
$$12x^3+24x^2+9x+2=0$$

Testing $x=-2$: $12(-8)+24(4)+9(-2)+2=-96+96-18+2=-16\neq 0$. Testing $x=-1$: $12(-1)+24(1)-9+2=5\neq 0$. By numerical inspection or the **Rational Root Theorem**, we find the root at $x\approx -1.565$. However, checking $x=-2/3$ (the zero of the numerator of f): $12(-8/27)+24(4/9)+9(-2/3)+2=-32/9+32/3-6+2=-3.55+10.66-4\neq 0$. Solving $12x^3+24x^2+9x+2=0$ numerically yields $x\approx -1.565$.

$\boxed{x\approx -1.565}$

4. Graph Sketching of $y=f(x)$

- **Intercepts**:
  - y-intercept: $f(0)=\frac{2}{-1}=-2$.
  - x-intercept: $3x+2=0 \implies x=-2/3\approx -0.667$.
- **Asymptotes**:
  - Vertical: $x=-0.5$ and $x=0.5$.
  - Horizontal: Since $\deg(\text{numerator}) < \deg(\text{denominator})$, $y=0$.
- **Extrema**: Set $f'(x)=0 \implies 12x^2+16x+3=0$.
  - $x=\frac{-16\pm\sqrt{256-144}}{24}=\frac{-16\pm\sqrt{112}}{24}=\frac{-16\pm4\sqrt{7}}{24}=\frac{-4\pm\sqrt{7}}{6}$.
  - $x_1\approx -1.108$ (Local Min), $x_2\approx -0.225$ (Local Max).



5. Asymptotes of $g(x)$

The function $g(x) = \frac{4x^2-1}{3x+2}$ is the reciprocal of $f(x)$.

- **Vertical Asymptote**: $3x + 2 = 0 \implies x = -2/3$.
- **Oblique Asymptote**: Perform polynomial long division.

$$\begin{aligned} 4x^2 - 1 &= \frac{4}{3}x(3x+2) - \frac{8}{3}x - 1 \\ &= \frac{4}{3}x(3x+2) - \frac{8}{9}(3x+2) + \frac{16}{9} - 1 \\ &= \left(\frac{4}{3}x - \frac{8}{9}\right)(3x+2) + \frac{7}{9} \end{aligned}$$

The **oblique asymptote** is $y = \frac{4}{3}x - \frac{8}{9}$.

$$\boxed{x = -2/3, y = \frac{4}{3}x - \frac{8}{9}}$$

6. Solving the Inequality $f(x) < g(x)$

We solve $f(x) < \frac{1}{f(x)}$. Let $y = f(x)$. We seek $y < \frac{1}{y}$. This is equivalent to $\frac{y^2-1}{y} < 0$, which implies $\frac{(y-1)(y+1)}{y} < 0$. The critical values for y are $-1, 0, 1$. The inequality holds when $y < -1$ or $0 < y < 1$.

1. $f(x) < -1 \implies \frac{3x+2}{4x^2-1} + 1 < 0 \implies \frac{4x^2+3x+1}{4x^2-1} < 0$. The numerator $4x^2 + 3x + 1$ has discriminant $\Delta = 9 - 16 = -7 < 0$, so it is always positive. Thus, we need $4x^2 - 1 < 0 \implies -0.5 < x < 0.5$.
2. $0 < f(x) < 1$:
   - $f(x) > 0 \implies \frac{3x+2}{(2x-1)(2x+1)} > 0$. Critical points: $-2/3, -1/2, 1/2$. Intervals: $(-2/3, -0.5) \cup (0.5, \infty)$.

- $f(x) < 1 \implies \frac{3x+2}{4x^2-1} - 1 < 0 \implies \frac{-4x^2+3x+3}{4x^2-1} < 0$. Roots of $-4x^2 + 3x + 3 = 0$: $x = \frac{-3\pm\sqrt{9+48}}{-8} = \frac{3\pm\sqrt{57}}{8} \approx -0.569, 1.319$. Combining these with the domain and $f(x) > 0$, we find the solution set.

$$\boxed{x \in (-2/3, -0.569) \cup (-0.5, 0.5) \cup (1.319, \infty)}$$

MathAA_21_HL_Summer_2021_Q12

Solution

1. Partial Fraction Decomposition

To express $f'(x) = \frac{1}{x(k-x)}$ in the form $\frac{a}{x} + \frac{b}{k-x}$, we use the method of **partial fraction decomposition**.

- Set up the identity:

$$\frac{1}{x(k-x)} = \frac{a}{x} + \frac{b}{k-x}$$

- Multiply both sides by the denominator $x(k-x)$:

$$1 = a(k-x) + bx$$

- To find a, let $x = 0$:

$$1 = a(k-0) \Longrightarrow a = \frac{1}{k}$$

- To find b, let $x = k$:

$$1 = b(k) \Longrightarrow b = \frac{1}{k}$$

Thus, the values are:

$$\boxed{a = \frac{1}{k}, \quad b = \frac{1}{k}}$$

2. Finding the Expression for $f(x)$

We integrate $f'(x)$ with respect to x:

$$\begin{aligned} f(x) &= \int \left(\frac{1/k}{x} + \frac{1/k}{k-x} \right) dx \\ &= \frac{1}{k} \int \frac{1}{x} dx + \frac{1}{k} \int \frac{1}{k-x} dx \\ &= \frac{1}{k} \ln|\ x\ | - \frac{1}{k} \ln|\ k-x\ | + C \\ &= \frac{1}{k} \ln\left| \frac{x}{k-x} \right| + C \end{aligned}$$

$$\boxed{f(x) = \frac{1}{k} \ln\left| \frac{x}{k-x} \right| + C}$$

3. Solving the Differential Equation for Population P

The **logistic growth model** is given by:

$$\frac{dP}{dt} = \frac{P(k-P)}{5k}$$

- Separate the variables:

$$\frac{1}{P(k-P)}dP = \frac{1}{5k}dt$$

- Integrate both sides using the result from part (a):

$$\int\left(\frac{1/k}{P} + \frac{1/k}{k-P}\right)dP = \int \frac{1}{5k}dt$$
$$\frac{1}{k}\ln\left|\frac{P}{k-P}\right| = \frac{1}{5k}t + C'$$

- Multiply by k:

$$\ln\left|\frac{P}{k-P}\right| = \frac{t}{5} + C''$$

- Exponentiate both sides:

$$\frac{P}{k-P} = Ae^{t/5}, \quad \text{where} \quad A = e^{C''}$$

- Use the initial condition $P(0) = 1200$:

$$A = \frac{1200}{k-1200}$$

- Substitute A back and solve for P:

$$\frac{P}{k-P} = \frac{1200}{k-1200}e^{t/5}$$
$$P(k-1200) = 1200(k-P)e^{t/5}$$
$$P(k-1200) + 1200Pe^{t/5} = 1200ke^{t/5}$$
$$P\left[(k-1200) + 1200e^{t/5}\right] = 1200ke^{t/5}$$
$$P = \frac{1200ke^{t/5}}{(k-1200) + 1200e^{t/5}}$$

- Divide numerator and denominator by $e^{t/5}$:

$$P = \frac{1200k}{(k-1200)e^{-t/5} + 1200}$$

4. Finding the Value of k

Given $P(10) = 2 \times 1200 = 2400$:

$$2400 = \frac{1200k}{(k-1200)e^{-2}+1200}$$
$$2 = \frac{k}{(k-1200)e^{-2}+1200}$$
$$2(k-1200)e^{-2}+2400 = k$$
$$2ke^{-2}-2400e^{-2}+2400 = k$$
$$k(2e^{-2}-1) = 2400e^{-2}-2400$$
$$k = \frac{2400(e^{-2}-1)}{2e^{-2}-1}$$

Using numerical calculation:

$$k \approx 2821.03...$$

$$\boxed{k \approx 2821}$$

5. Time of Maximum Rate of Change

The rate of change $\frac{dP}{dt}$ is a quadratic function of P: $R(P) = \frac{1}{5k}(kP - P^2)$. The maximum occurs at the vertex of the parabola, which is the **inflection point** of the logistic curve.

- The maximum rate occurs when $P = \frac{k}{2}$.
- Substitute $P = \frac{k}{2}$ into the expression for $\frac{P}{k-P}$:

$$\frac{k/2}{k-k/2} = \frac{1200}{k-1200}e^{t/5}$$
$$1 = \frac{1200}{k-1200}e^{t/5}$$
$$e^{t/5} = \frac{k-1200}{1200}$$
$$t = 5\ln\left(\frac{k-1200}{1200}\right)$$

- Substitute $k \approx 2821.03$:

$$t \approx 5\ln\left(\frac{2821.03-1200}{1200}\right) \approx 5\ln(1.3508) \approx 1.503$$

[Visualization]

$$\boxed{t \approx 1.50\,\text{days}}$$